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Invariant sets in the Clebsch–Tisserand problem: Existence and stability☆

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Abstract

Questions of the existence and stability of invariant sets in the problem of the motion of a rigid body with a fixed point in an axi-symmetric force field with a quadratic potential (with respect to the direction cosines of the axis of symmetry of the field) are discussed. This problem is isomorphic with the problem of the motion of a free rigid body bounded by a simply connected surface in an ideal homogeneous incompressible fluid which performs irrotational motion and is at rest at infinity. [Kirchhoff F G.R. Über die Bewegung eines Rotationskörpers in eine Flüssigkeit, J Reine und angew Math, B. 71, 1870, S.237–S.262; Clebsch A. Über die Bewegung eines Köipers in eine Flüssigkeit, Math Anmnalen, Bd. 3, 1870, S.238–S.262] In particular, in the second case of the Clebsch integrability of a problem of the motion of a body in a fluid, this problem is isomorphic with the Tisserand problem on the motion of a body with a fixed point in the case of a quadratic potential of special form,[Tisserand M.F. Sur le mouvement des planètes autour du Soleil d'apreès la lol èlectrodynamique de Weber, C.R. Acad. Sci. Paris, 1872. V. 75, P. 760–763] which corresponds to the satellite approximation of the potential of the forces of Newtonian attraction.

1. Formulation of the problem[1,3]

Consider a rigid body with a fixed point in an axi-symmetric force field with a quadratic potential corresponding to the second Clebsch case.[2] The kinetic energy of the body and the potential have the form

$$T = \frac{1}{2}\sum_{(123)} A_1\omega_1^2, \quad V = \frac{1}{2}\kappa^2\sum_{(123)} A_1\gamma_1^2$$

Here, $A_i(A_1 < A_2 < A_3)$ are the principal moments of inertia of the body, ω_i and γ_i are the components of the angular velocity vector and the unit vector of the axis of symmetry of the field in the principal axes of inertia of the body for the fixed point, κ^2 is a constant which, without loss of generality, we shall consider to be equal to unity by choosing the appropriate unit of time, $i = 1, 2, 3$ and the symbol (123) denotes cyclic permutation of the subscripts 1, 2, and 3.

The equations of motion of the body in Euler–Poisson form

$$A_1\dot{\omega}_1 + (A_3 - A_2)(\omega_2\omega_3 - \gamma_2\gamma_3) = 0, \quad \dot{\gamma}_1 + \omega_2\gamma_3 - \omega_3\gamma_2 = 0 \ (123) \tag{1.1}$$

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admit of the four first integrals

$$H = \frac{1}{2}\sum_{(123)} A_1(\omega_1^2 + \gamma_1^2) = h = \text{const}, \quad K = \sum_{(123)} A_1\omega_1\gamma_1 = k = \text{const}, \ \Gamma = \sum_{(123)} \gamma_1^2 = 1 \tag{1.2}$$

$$C = \sum_{(123)} (A_1^2\omega_1^2 - A_2A_3\gamma_1^2) = c = \text{const} \tag{1.3}$$

(energy, area, geometry and Clebsch; compare with Ref. 4).

2. Invariant sets

According to the modified Routh theorem,[5] the critical sets of one of the first integrals (1.2) and (1.3) at fixed levels of the other first integrals correspond to the invariant sets of systems (1.1). We shall seek the critical sets of the integral C at fixed levels of the integrals

$$H = h, K = k, \Gamma = 1$$

To do this, we introduce the function

$$2W = C - \lambda(H - h) - 2\mu(K - k) + \nu(\Gamma - 1)$$

where γ, μ and ν are Lagrange undetermined multipliers and we write the condition for its stationary state as

$$\frac{\partial W}{\partial \omega_1} = A_1[(A_1 - \lambda)\omega_1 - \mu\gamma_1] = 0 \ (123) \tag{2.1}$$

$$\frac{\partial W}{\partial \gamma_1} = (\nu - \lambda A_1 - A_2A_3)\gamma_1 - \mu A_1\omega_1 = 0 \ (123) \tag{2.2}$$

Equations (1.2), which are the conditions for the stationary state of the function W with respect to the undetermined multipliers, has to be added to Eqs. (2.1) and (2.2).

Assuming that $\lambda \neq 0, A_1, A_2$ or A_3, we find the values of ω_i from Eq. (2.1) and, substituting them into Eq. (2.2) assuming that $\gamma_i \neq 0$ $(i = 1, 2, 3)$, we conclude that

$$\nu = \lambda A_1 + A_2A_3 + A_1(A_1 - \lambda)^{-1}\mu^2 \ (123)$$

where it follows that

$$\mu^2 = (\lambda - A_1)(\lambda - A_2)(\lambda - A_3)\lambda^{-1} \tag{2.3}$$

Hence, if

$$\lambda \in (-\infty, 0) \cup (A_1, A_2) \cup (A_3, +\infty) \tag{2.4}$$

(we recall that $A_1 < A_2 < A_3$), then, for fixed levels of integrals (1.2), the integral (1.3) takes a steady value in the two-dimensional sets of the form

$$\omega_i = \lambda_i\gamma_i, \quad i = 1, 2, 3, \quad (\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1) \tag{2.5}$$

where $\lambda_i = \mu(A_i - \lambda)^{-1}$ and $\mu = \mu(\lambda)$ is defined by relation (2.3). It is obvious that, if $\lambda < 0$ or $\lambda > A_3$, then all the λ_i have the same sign (which corresponds to the sign of μ or its opposite respectively), but, if $\lambda \in (A_1, A_2)$, then λ_2 and λ_3 have the same sign which is identical to the sign of μ, and λ_1 has the opposite sign. Without loss of generality, we will now assume that $\mu > 0$ (the case when $\mu < 0$ is obtained from the preceding case by replacing k by $-k$).

In the general case, the two-dimensional sets (2.5) are parametrized by the quantity λ which, of course, depends on the constants h and k of the energy and area integrals. Actually, substituting relations (2.5) into the expressions for the first two integrals of (2.5) and taking account of the third integral, we obtain

$$\sum_{(123)} A_1\left[1+\frac{\mu^2}{(\lambda-A_1)^2}\right]\gamma_1^2 = 2h, \quad \sum_{(123)} A_1\frac{\mu}{\lambda-A_1}\gamma_1^2 = -k, \quad \sum_{(123)}\gamma_1^2 = 1 \tag{2.6}$$

Expressing γ_1^2 and γ_2^2 from the last two relations of (2.6) in terms of γ_3^2 and k and substituting the result into the first equality of (2.6), we can show that the coefficient of γ_3^2 identically vanishes. At the same time, the first relation of (2.6) takes the form

$$2h = \frac{\tilde{A}}{\lambda^2} - \frac{k}{\mu\lambda^2}[2\lambda^3 - (A_1+A_2+A_3)\lambda^2 + \tilde{A}]; \quad \tilde{A} = A_1A_2A_3 \tag{2.7}$$

Hence, the invariant sets (2.5) form two-parameter families (condition (2.4) must not be omitted here).

3. Dynamics of a body on invariant sets

Substituting expressions (2.5) into the last system of equations (1.1), we obtain

$$\dot{\gamma}_1 + (\lambda_2-\lambda_3)\gamma_2\gamma_3 = 0 \ (123) \tag{3.1}$$

(the first system of equations (1.1) is satisfied identically in this case).

It is obvious that Eq. (3.1) are isomorphic with the equations of motion of the Euler top and admit of the two integrals

$$\Gamma = \sum_{(123)}\gamma_1^2 = 1 \tag{3.2}$$

$$L = \sum_{(123)}\lambda_1\gamma_1^2 = l = \frac{k-\mu}{\lambda} = l(h,k) = \text{const} \tag{3.3}$$

(the value of the constant l is easily obtained by eliminating the variables γ_1^2 and γ_2^2 from the expression for L using the first two relations of (2.6) and taking account of relation (2.3) and (2.7)). Note that, in the analysis of Eq. (3.1), instead of the integral (3.3), it is more convenient to use the integral

$$K_0 = \sum_{(123)} A_1\lambda_1\gamma_1^2 = k \tag{3.4}$$

which is obtained from the integrals (3.2) and (3.3) ($K_0 = \mu\Gamma + \lambda L$); its constant is identical to the constant k of the initial area integral. However, integral (3.4) depends, of course, on the constant h of the initial energy integral, since the constants λ_i which depend on λ, that is, on k and h, occur on the left-hand side of relation (3.4).

Hence the motion of a body on the invariant sets is described by the elliptic functions of time

$$\omega_i = \lambda_i\gamma_i^0(t) \equiv \omega_i^0(t), \quad \gamma_i = \gamma_i^0(t), \quad i = 1, 2, 3$$

where $\gamma_i^0(t)(i = 1,\ 2,\ 3)$ is the general solution of system (3.1). We note that, unlike in the classical Euler-Poinsot problem, for which the surfaces of the levels of its integrals are always ellipsoids, this only holds in problem (3.1) when $\lambda < 0$ or $\lambda > A_3$; when $\lambda \in (A_1, A_2)$, the surfaces for a level of the integral (3.4) when $k \neq 0$ are two-sheeted (when $k\mu < 0$) or one-sheeted (when $k\mu > 0$) hyperboloids and, when $k = 0$, they are a cone.

Nevertheless, the combined levels of the first integrals (3.2) and (3.4) define (when account is taken of relations (2.5)) the one-dimensional invariant sets of system (1.1) for any admissible values of the constants h and k; when $2h = A_i + k^2/A_i$ $(i = 1, 2, 3)$, they degenerate into zero-dimensional sets which correspond to the permanent rotations of the body around the principal axes of inertia.

4. The stability of the invariant sets

We will now calculate the second variation of the function W

$$2\delta^2 W = -\sum_{(123)} \tilde{A}_1 u_1^2; \quad u_1 = \delta(\omega_1 - \lambda_1\gamma_1), \quad \tilde{A}_1 = A_1(\lambda - A_1)\ (123) \tag{4.1}$$

According to the modified Routh theorem, the invariant sets (2.5) and (3.4) are stable if the quadratic form (4.1) is sign-definite in the linear manifold $\delta H = \delta K = \delta\Gamma = 0$, which is defined by the relations

$$\sum_{(123)} A_1\gamma_1^0(t)(\lambda_1\delta\omega_1 + \delta\gamma_1) = 0, \quad \sum_{(123)} A_1\gamma_1^0(t)(\delta\omega_1 + \lambda_1\delta\gamma_1) = 0, \quad \sum_{(123)} \gamma_1^0(t)\delta\gamma_1 = 0 \tag{4.2}$$

Expressing $\gamma_1^0(t)\delta\gamma_1$ and $\gamma_2^0(t)\delta\gamma_2$ from the last two relations of (4.2) in terms of $\gamma_3^0(t)\delta\gamma_3$ and $\delta\omega_1$, $\delta\omega_2$, $\delta\omega_3$ and substituting the resulting expressions into the first relation of (4.2), we find a relation which solely contains u_1, u_2, u_3:

$$\sum_{(123)} B_1 u_1 = 0; \quad B_1 = A_1\tilde{B}_1\gamma_1^0(t), \quad \tilde{B}_1 = (A_3 + A_2 - A_1)\lambda^2 - 2A_2A_3\lambda + \tilde{A} \tag{4.3}$$

It is obvious that, if $\lambda < 0$ or $\lambda > A_3$, then the quadratic form (4.1) is sign-definite for any u_1, u_2, u_3 (in particular, when u_1, u_2, u_3, which satisfy relation (4.3)). Consequently, the invariant sets (2.5) and (3.4) corresponding to $\lambda < 0$ or $\lambda > A_3$ are stable, since the extremal values yield the integral (1.3) at fixed levels of the integral (1.2) (see also Ref. 4).

If $\lambda \in (A_1, A_2)$, the quadratic form (4.1) is not sign-definite for arbitrary u_1, u_2, u_3. We will now show that, depending on the parameters of the problem, the quadratic form (4.1) either remains not sign-definite (the invariant sets are unstable in this case[5]) or becomes sign-definite in the manifold (4.3) (the invariant sets are stable in this case). To do this, we calculate the determinant

$$\Delta = -\begin{vmatrix} 0 & B_1 & B_2 & B_3 \\ B_1 & -\tilde{A}_1 & 0 & 0 \\ B_2 & 0 & -\tilde{A}_2 & 0 \\ B_3 & 0 & 0 & -\tilde{A}_3 \end{vmatrix} = \tilde{A}\sum_{(123)} (\lambda - A_1)(\lambda - A_2)A_3\tilde{B}_3^2(\gamma_3^0(t))^2 \tag{4.4}$$

We recall that the functions $\gamma_i^0(t)(i = 1, 2, 3)$ satisfy system (3.1) and, consequently, relations (3.2) and (3.4). Expressing $\gamma_1^0(t)$ and $\gamma_2^0(t)$ from these latter relations in terms of $\gamma_3^0(t)$ and k and substituting the resulting expressions into equality (4.4), we find that the coefficient of $(\gamma_3^0(t))^2$ vanishes and, as a result of this,

$$\Delta = \tilde{A}[4\tilde{A}(\lambda - A_1)^2(\lambda - A_2)^2(\lambda - A_3)^2 - k\mu\lambda P(\lambda)]$$
$$P(\lambda) = \left[\sum_{(123)} (A_1^2 - 2A_2A_3)\right]\lambda^4 + \tilde{A}\left[12\lambda^3 - 6\lambda^2\sum_{(123)} A_1 + 4\lambda\sum_{(123)} A_2A_3 - 3\tilde{A}\right] \tag{4.5}$$

Here, $P(\lambda) > P_0 > 0$ for all $\lambda \in (A_1, A_2)$. In fact, we consider the function $Q(1/\lambda) = P(\lambda)/\lambda^4$. If is obvious that

$$dQ/d(1/\lambda) = 12\tilde{A}(\lambda - A_1)(\lambda - A_2)(\lambda - A_3)/\lambda^3 > 0, \quad \forall\lambda \in (A_1, A_2)$$

Consequently, $Q(1/\lambda) > Q(1/A_2)$, that is,

$$P(\lambda) > (A_1/A_2)^4 P(A_2) = A_1^4(A_2 - A_1)^2(A_2 - A_3)^2/A_2^2 > 0$$

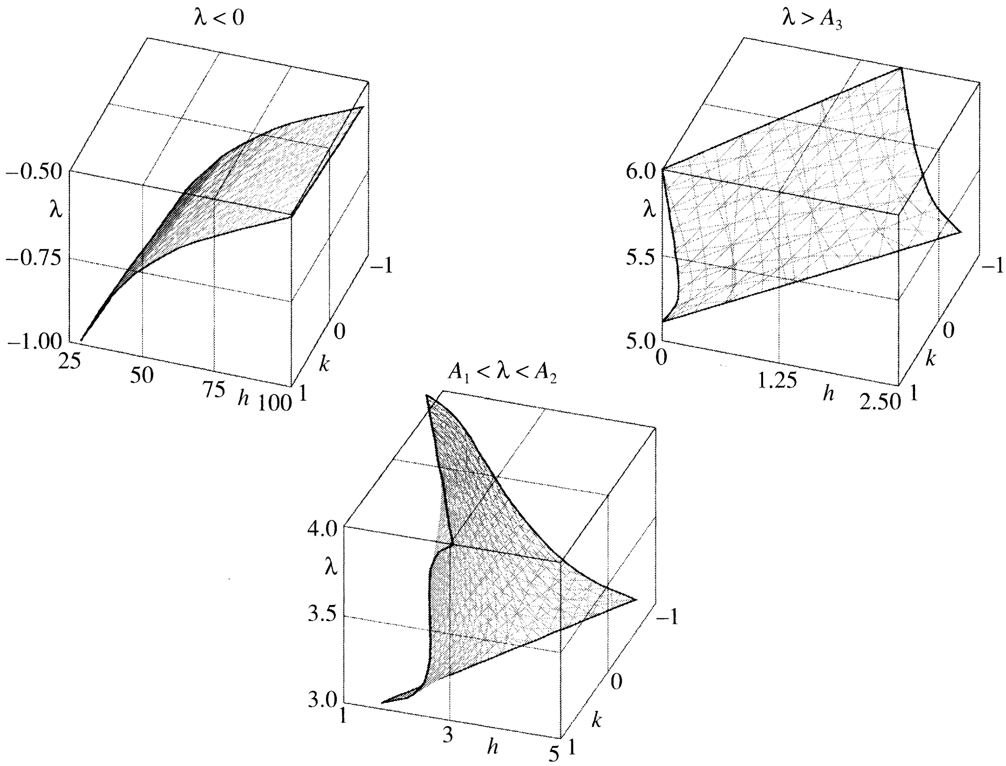

Fig. 1.

Hence (we recall that $\mu > 0$, $\lambda \in (A_1, A_2)$), the determinant (4.5) is greater than zero for all $k \leq 0$ and less than zero for all $k \geq k_0$, where k_0 is the maximum of the function

$$R(\lambda) = 4\tilde{A}(\lambda - A_1)(\lambda - A_2)(\lambda - A_3)\mu(\lambda)/P(\lambda)$$

in the interval $\lambda \in [A_1, A_2]$ and always changes sign in the interval $\lambda \in (A_1, A_2)$ when $k \in (0, k_0)$.

5. Geometrical interpretation

In the space (h, k, λ), the invariant sets (2.5) and (3.4) correspond to a surface $\lambda = \lambda(h, k)$ defined by relation (2.7). It is obvious that, for any fixed value of k,

$$dh/d\lambda = -\tilde{A}/\lambda^3 + k\mu P(\lambda)/[4\lambda^2(\lambda - A_1)^2(\lambda - A_2)^2(\lambda - A_3)^2]$$

Consequently (see expression (4.5)),

$$\partial\lambda/\partial h = -4A_1A_2A_3\lambda^3(\lambda - A_1)^2(\lambda - A_2)^2(\lambda - A_3)^2/\Delta$$

Hence, the segments of the surface (2.7) for which $\lambda < 0$ or $\lambda > A_3$ and, also, those of the segments of this surface conforming to $\lambda \in (A_1, A_2)$, for which the function $\lambda(h, k)$ decreases as increases h in the case of fixed k, correspond to stable invariant sets (see Fig. 1). Segments of the surface (2.7) conforming to $\lambda \in (A_1, A_2)$ for which the function $\lambda(h, k)$ increases as h increases in the case of fixed k, correspond to unstable invariant sets.

In conclusion, we note that, if the integral

$$G = \rho\Gamma + \sigma L = g = \text{const}$$

where $g=-\Delta/(4\tilde{A}\mu^4\lambda^2)=g(\lambda(h,\ k)),\ \sigma=P(\lambda)/(4\tilde{A}\mu^2),\ \rho=\sigma\mu\lambda^{-1}$ is introduced instead of the integral K_0, then the stable (unstable) invariant sets conforming to $\lambda\in(A_1,\ A_2)$ correspond to the case when the surface $G=g$ is a two-sheeted (one-sheeted) hyperboloid.

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